

[...]

2. A Lanczos-type MIMO method obtained as Block-Arnoldi generalization.

Briefly:

Having as a starting point the MIMO descriptor system

$$\begin{cases} \mathbf{C}\dot{\mathbf{x}}(t) = -\mathbf{G}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{L}^T \mathbf{x}(t) \end{cases} \quad \text{respectively} \quad \begin{cases} s\mathbf{C}\mathbf{x}(s) = -\mathbf{G}\mathbf{x}(s) + \mathbf{B}\mathbf{u}(s) \\ \mathbf{y}(s) = \mathbf{L}^T \mathbf{x}(s) \end{cases} \quad (6)$$

(where the dimension of the matrices are: $\mathbf{C}, \mathbf{G} \in \mathbf{R}^{N \times N}$, $\mathbf{B} \in \mathbf{R}^{N \times m}$, $\mathbf{L} \in \mathbf{R}^{N \times m}$, \mathbf{B} and \mathbf{L} are full rank matrices), Lanczos type MIMO method sets up two biorthonormal bases $\mathbf{V}=[\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_k]$, $\mathbf{W}=[\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_k]$ which determine in each iteration \mathbf{k} a reduced \mathbf{n} -dimensional ($\mathbf{n}=\mathbf{k}\mathbf{m}$) model of the initial system:

$$\begin{cases} \mathbf{C}_n \dot{\mathbf{x}}(t) = -\mathbf{G}_n \mathbf{x}(t) + \mathbf{B}_n \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{L}_n^T \mathbf{x}(t) \end{cases} \quad \text{respectively} \quad \begin{cases} s\mathbf{C}_n \mathbf{x}(s) = -\mathbf{G}_n \mathbf{x}(s) + \mathbf{B}_n \mathbf{u}(s) \\ \mathbf{y}(s) = \mathbf{L}_n^T \mathbf{x}(s) \end{cases} \quad (7)$$

where $\mathbf{G}_n = \mathbf{W}^T \mathbf{G} \mathbf{V}$, $\mathbf{C}_n = \mathbf{W}^T \mathbf{C} \mathbf{V}$, $\mathbf{B}_n = \mathbf{W}^T \mathbf{B}$, $\mathbf{L}_n = \mathbf{V}^T \mathbf{L}$.

2.1. Markov moments expressions for MIMO systems.

Consider:

i) $s_0 \in \mathbf{C}$ such that the matrix $(\mathbf{G} + s_0 \mathbf{C})$ is nonsingular.

ii) $\mathbf{F}_1, \mathbf{F}_2 \in \mathbf{C}^{N \times N}$ two nonsingular matrices that form a factorization of the matrix $\mathbf{G} + s_0 \mathbf{C}$, i.e. $\mathbf{G} + s_0 \mathbf{C} = \mathbf{F}_1 \mathbf{F}_2$.

iii) $\mathbf{H}(s) = \sum_0^{\infty} \mathbf{M}_i (s - s_0)^i$ the Taylor expansion of the transfer function of the

initial system (7) around the complex number s_0 .

Let us denote: $\mathbf{A} = -\mathbf{F}_1^{-1} \mathbf{C} \mathbf{F}_2^{-1}$; $\mathbf{R} = \mathbf{F}_1^{-1} \mathbf{B}$; $\mathbf{L} = \mathbf{F}_2^{-T} \mathbf{E}$; $s = s_0 + \sigma$.

Hence $\mathbf{H}(s)$ can be rewritten as:

$$\mathbf{H}(s) = \mathbf{L}^T (\mathbf{I} - (s - s_0) \mathbf{A})^{-1} \mathbf{R} = \mathbf{L}^T (\mathbf{I} - \sigma \mathbf{A})^{-1} \mathbf{R}$$

and consequently

$$\mathbf{H}(s) = \mathbf{L}^T \left(\sum_{i=0}^{\infty} \mathbf{A}^i (s - s_0)^i \right) \mathbf{R} = \sum_{i=0}^{\infty} \mathbf{L}^T \mathbf{A}^i \mathbf{R} \sigma^i,$$

where the matricial coefficients $\mathbf{M}_i = \mathbf{L}^T \mathbf{A}^i \mathbf{R}$ are Markov moments. The relation above proves that describing the input-output behaviours of system (6) around

$s=s_0$ is equivalent with describing the input-output behavior of the system

$$\begin{cases} (\mathbf{I} - \sigma\mathbf{A})\mathbf{x}(\sigma) = \mathbf{R}\mathbf{u}(\sigma) \\ \mathbf{y}(\sigma) = \mathbf{L}^T \mathbf{x}(\sigma) \end{cases} \text{ around } s=0.$$

2.2. The algorithm.

Consider \mathbf{A} , \mathbf{R} , \mathbf{L} as defined above with $\mathbf{F}_1\mathbf{F}_2$ being the trivial factorization of the nonsingular pencil $\mathbf{G}+s_0\mathbf{C}$, i.e. $\mathbf{F}_1\mathbf{F}_2 = (\mathbf{G}+s_0\mathbf{C})\mathbf{I}_N$ and $\mathbf{K}_k(\mathbf{A},\mathbf{R})=\text{span}\{\mathbf{R}, \mathbf{A}\mathbf{R}, \mathbf{A}^2\mathbf{R}, \dots, \mathbf{A}^{k-1}\mathbf{R}\}$, $\mathbf{K}_k(\mathbf{A}^T,\mathbf{L})=\text{span}\{\mathbf{L}, \mathbf{A}^T\mathbf{L}, (\mathbf{A}^T)^2\mathbf{L}, \dots, (\mathbf{A}^T)^{k-1}\mathbf{L}\}$ being k -order Krylov subspaces generated by \mathbf{A} and \mathbf{R} , respectively by \mathbf{A}^T and \mathbf{L} . The following iterative algorithm sets up the biorthonormal bases \mathbf{V} and \mathbf{W} in the right and left Krylov subspaces $\mathbf{K}_k(\mathbf{A},\mathbf{R})$ respectively $\mathbf{K}_k(\mathbf{A}^T, \mathbf{L})$:

1. $\mathbf{V}_1^2 = \mathbf{R}$; $\mathbf{W}_1^2 = \mathbf{L}$; $\mathbf{V}_0 = 0$; $\mathbf{W}_0 = 0$; $\mathbf{D}_1 = 0$;
2. Compute QR¹ decomposition of \mathbf{V}_1^2 and \mathbf{W}_1^2 :
 $\mathbf{V}_1^2 = \mathbf{V}_1^1\mathbf{D}_1^1$; $\mathbf{W}_1^2 = \mathbf{W}_1^1\mathbf{B}_1^1$;
3. Compute SVD² of $(\mathbf{W}_1^1)^T \mathbf{V}_1^1 = \mathbf{P}_1\mathbf{S}_1\mathbf{Q}_1^T$;
4. $\mathbf{V}_1 = \mathbf{V}_1^1\mathbf{Q}_1\mathbf{S}_1^{-\frac{1}{2}}$; $\mathbf{W}_1 = \mathbf{W}_1^1\mathbf{P}_1^T\mathbf{S}_1^{-\frac{1}{2}}$;
 $\mathbf{C}_1 = \mathbf{W}_1^T \mathbf{A}\mathbf{V}_1$;
5. For $j=1, \dots, k$ Do :
 6. $\mathbf{V}_{j+1}^2 = \mathbf{A}\mathbf{V}_j - \mathbf{V}_j\mathbf{C}_j - \mathbf{V}_{j-1}\mathbf{B}_j$;
 $\mathbf{W}_{j+1}^2 = \mathbf{A}^T\mathbf{W}_j - \mathbf{W}_j\mathbf{C}_j^T - \mathbf{W}_{j-1}\mathbf{D}_j^T$;
 7. Compute QR decomposition of \mathbf{V}_{j+1}^2 and \mathbf{W}_{j+1}^2 :
 $\mathbf{V}_{j+1}^2 = \mathbf{V}_{j+1}^1\mathbf{D}_{j+1}^1$; $\mathbf{W}_{j+1}^2 = \mathbf{W}_{j+1}^1(\mathbf{B}_{j+1}^1)^T$;
 8. Compute SVD of $(\mathbf{W}_{j+1}^1)^T \mathbf{V}_{j+1}^1 = \mathbf{P}_{j+1}\mathbf{S}_{j+1}\mathbf{Q}_{j+1}^T$;

¹ QR Decomposition – B. Dumitrescu, C. Popeea, B. Jora; *Metode de calcul numeric matriceal. Algoritmi fundamentali*, All Educational, 1998, Bucharest, p. 150, Theorem 3.2.

² Singular Values Decomposition – B. Dumitrescu, C. Popeea, B. Jora; *Metode de calcul numeric matriceal. Algoritmi fundamentali*, All Educational, 1998, Bucharest, p.370, Theorem 5.1.

$$\begin{aligned}
 \text{i) } & \mathbf{V}\mathbf{W}^T\mathbf{V}\mathbf{W}^T = \mathbf{V}\mathbf{W}^T \quad \text{i.e. } \mathbf{P}^2 = \mathbf{P}, \\
 \text{ii) } & \forall \mathbf{v} \in \mathbf{K}_k(\mathbf{A}, \mathbf{r}) : \mathbf{V}\mathbf{W}^T\mathbf{v} = \mathbf{v} \quad \text{i.e. } \mathbf{P}|_{\mathbf{K}_k(\mathbf{A}, \mathbf{r})} = \mathbf{I}_N, \\
 \text{iii) } & \forall \overline{\mathbf{w}} \in \mathbf{K}_k^\perp(\mathbf{A}^T, \mathbf{l}) : \mathbf{V}\mathbf{W}^T\overline{\mathbf{w}} = \mathbf{0} \quad \text{i.e. } \mathbf{P}|_{\mathbf{K}_k^\perp(\mathbf{A}^T, \mathbf{l})} = \mathbf{O}_n.
 \end{aligned} \tag{9}$$

Therefore $\mathbf{V}\mathbf{W}^T$ is the projector from \mathbf{R}^N on $\mathbf{K}_k(\mathbf{A}, \mathbf{r})$ orthogonal to $\mathbf{K}_k(\mathbf{A}^T, \mathbf{l})$.

2.3. The reduced order model realization through the Lanczos-type MIMO algorithm.

The following will demonstrate that the relations (8),(9) in themselves determine an n -dimensional ($n=km$) reduced model of the initial system (6).

$$\begin{aligned}
 \mathbf{W}^T\mathbf{A}\mathbf{V} = \mathbf{T}_k & \Rightarrow -\mathbf{W}^T(\mathbf{G} + s_0\mathbf{C})^{-1}\mathbf{C}\mathbf{V} = \mathbf{T}_k \Rightarrow \\
 -\mathbf{W}^T(\mathbf{G} + s_0\mathbf{C})^{-1}\mathbf{V}\mathbf{W}^T\mathbf{C}\mathbf{V} & = \mathbf{T}_k \Rightarrow -(\mathbf{W}^T(\mathbf{G} + s_0\mathbf{C})\mathbf{V})^{-1}\mathbf{W}^T\mathbf{C}\mathbf{V} = \mathbf{T}_k \Rightarrow \\
 -(\mathbf{W}^T\mathbf{G}\mathbf{V} + s_0\mathbf{W}^T\mathbf{C}\mathbf{V})^{-1}\mathbf{W}^T\mathbf{C}\mathbf{V} & = \mathbf{T}_k \Rightarrow \mathbf{A}_n = -(\mathbf{G}_n + s_0\mathbf{C}_n)^{-1}\mathbf{C}_n = \mathbf{T}_k
 \end{aligned}$$

where $\mathbf{G}_n = \mathbf{W}^T\mathbf{G}\mathbf{V}$ and $\mathbf{C}_n = \mathbf{W}^T\mathbf{C}\mathbf{V}$.

Moreover, $\forall j \in \overline{0, 2n-1}$:

$$\begin{aligned}
 \mathbf{M}_j & = \mathbf{L}^T\mathbf{A}^j\mathbf{R} = \mathbf{L}^T\mathbf{V}\mathbf{W}^T\mathbf{A}^j\mathbf{V}\mathbf{W}^T\mathbf{R} = (\mathbf{V}^T\mathbf{L})^T\mathbf{T}_k^j\mathbf{W}^T\mathbf{R} = \\
 & = \mathbf{L}_n^T\mathbf{T}_k^j\mathbf{W}^T(\mathbf{G} + s_0\mathbf{C})^{-1}\mathbf{V}\mathbf{W}^T\mathbf{B} = \mathbf{L}_n^T\mathbf{T}_k^j(\mathbf{W}^T(\mathbf{G} + s_0\mathbf{C})\mathbf{V})^{-1}(\mathbf{W}^T\mathbf{B}) = \\
 & = \mathbf{L}_n^T\mathbf{T}_k^j(\mathbf{W}^T\mathbf{G}\mathbf{V} + s_0\mathbf{W}^T\mathbf{C}\mathbf{V})^{-1}(\mathbf{W}^T\mathbf{B}) = \mathbf{L}_n^T\mathbf{T}_k^j(\mathbf{G}_n + s_0\mathbf{C}_n)^{-1}\mathbf{B}_n = \\
 & = \mathbf{L}_n^T\mathbf{A}_n^j\mathbf{R}_n = \mathbf{M}_{n,j},
 \end{aligned}$$

where $\mathbf{G}_n = \mathbf{W}^T\mathbf{G}\mathbf{V}$, $\mathbf{C}_n = \mathbf{W}^T\mathbf{C}\mathbf{V}$, $\mathbf{B}_n = \mathbf{W}^T\mathbf{B}$, $\mathbf{L}_n = \mathbf{V}^T\mathbf{L}$.

Hence we proved the following theorem:

Theorem 2. (Padé via Lanczos-type MIMO algorithm) The setting up of the matrices \mathbf{V} , \mathbf{W} , by which the Lanczos-type MIMO iterative process applied to the initial system (6) results in a reduced n -dimensional ($n=km$) model (7) after k iterations, is equivalent with finding a local $2n$ -order Padé approximation of the transfer function of the initial system, and also with an oblique projection of the initial system on $\mathbf{K}_n(\mathbf{A}, \mathbf{R})$ along the subspace $\mathbf{K}_n^\perp(\mathbf{A}^T, \mathbf{L})$. [...]